CERN-TH/2000-139

LARGE-N EXPANSION AND CONFORMAL WINDOWS IN THREE DIMENSIONS

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Abstract

I study a class of interacting conformal field theories and conformal windows in three dimensions, formulated using the Parisi large-N approach. Bosons are associated with composite operators and their propagators are dynamically generated by fermion bubbles. Renormalization-group flows between pairs of interacting fixed points satisfy a set of non-perturbative $g \leftrightarrow 1/g$ dualities. An exact relation between the beta function and the anomalous dimension of the composite boson can be written.

CERN-TH/2000-139 – May, 2000

Unexpected remnants of the renormalization algorithm in quantum field theory are the Adler–Bell–Jackiw anomalies, finite amplitudes arising from the quantum violation of classical conservation laws. Anomalies fall into two main classes: axial and trace. The axial anomalies obey all-order properties, such as the Adler–Bardeen theorem and give important information about the low-energy physics, by means of the 't Hooft anomaly-matching conditions. The trace anomaly is related to the beta function, by a formula $\Theta = \beta^a \mathcal{O}^a$, $\mathcal{O}^a$ being composite operators. Certain trace anomalies in external fields can be computed exactly in the IR limit of (supersymmetric) UV-free theories [1, 2], where an exact beta function can also be derived [3]. They reveal the intrinsic irreversibility of the renormalization-group flow, its relation to the invariant area of the graph of the beta function between the fixed points [4] and the essential difference between marginal and relevant deformations [5].

Most of these powerful results apply only to even dimensions. Trace anomalies in external gravitational and flavour fields do not exist in odd dimensions. Nevertheless, an odd-dimensional formula for the irreversibility of the RG flow can in principle be written [6], because the relation $\Theta = \beta^a \mathcal{O}^a$ is completely general and so is the notion of invariant area of the graph of the beta function. A web of non-trivial conformal field theories, conformal windows and RG flows in three dimensions would be useful to investigate these and related issues more closely. Here I address the construction of a large class of such theories and flows, following ref. [7].

The beta functions of the most general power-counting renormalizable three-dimensional theory with a Chern–Simons vector field have been studied by Avdeev *et al.* in ref. [8]. Three-dimensional quantum field theory is relevant for its possible applications in the domain of condensed-matter physics. However, the Chern–Simons models are parity-violating and this somewhat limits the range of their applicability. The three-dimensional ϕ^6 theory is known to have a non-trivial fixed point in the large-N expansion and a conformal window interpolating between the free limit and this point [9]. Nevertheless, a large class of parity-preserving conformal windows is not known at present and will be constructed here.

The Chern–Simons coupling $g_{\rm cs}$ is not renormalized [10, 11, 12]. The simplest argument to prove this fact proceeds as follows. Let us denote by $\beta_{\rm cs}$ the beta function of $g_{\rm cs}$. The relation between the trace anomaly and the beta function implies that Θ should contain a term proportional to the Chern–Simons form, multiplied by $\beta_{\rm cs}$. However, Θ is gauge-invariant, while the Chern–Simons form is not. For this reason $\beta_{\rm cs}$ has to be identically zero.

It was shown in ref. [8] that the Chern–Simons coupling can be used to split the zeros of the beta function and generate a variety of non-trivial conformal windows. For example, the beta function of a $\bar{\varphi}\varphi\bar{\psi}\psi$-coupling with constant η typically reads

$$\beta_\eta = a(\eta + b g_{\rm cs}^2)(\eta^2 - g_{\rm cs}^4)$$

to the lowest order, a and b being some factors, possibily depending on the gauge group and the representation. The coupling $g_{\rm cs}$ is quantized in the non-Abelian case, $g_{\rm cs}^2 = 1/N$, and is arbitrarily small in the large-N limit. Therefore, the existence of interacting fixed points at $\eta = \pm g_{\rm cs}^2$ and $\eta = -b g_{\rm cs}^2$ is proved, in this limit, directly from perturbation theory. This construction is a three-dimensional analogue of the existence proof of a conformal window in

QCD. There we have, to two loops,

$$\beta_{\rm QCD}=\beta_1\alpha^2+\beta_2\alpha^3+\mathcal{O}(\alpha^4),\qquad \beta_1=-\frac{1}{6\pi}(11N_c-2N_f),\qquad \beta_2=\frac{25N_c^2}{(4\pi)^2},$$

where β_2 is written for $\beta_1\ll N_c$ and N_c large. The role of $g_{\rm cs}$ is here played by $\beta_1/\beta_2\ll 1/N_c$. We see that all these constructions involve a large-N limit of some sort. Our models will not be an exception in this respect.

The successful removal of divergences in quantum field theory is not restricted to the power-counting renormalizable theories. Non-renormalizable models in less than four dimensions can be quantized using a large-N expansion [13]. The four-fermion model has been studied in detail [14, 15], and the technique has been applied to other cases, such as the S_{N-1} non-linear σ-model [16] and the CP^{N-1} model [17]. Here I use this technique to construct new non-trivial conformal windows.

I work in the Euclidean framework. The naive dimensional technique is not applicable and it is necessary to add a peculiar non-local term $\mathcal{L}_{\rm non\,loc}$ to the classical lagrangian. Details can be found in ref. [7].

The four-fermion model, written in terms of an auxiliary field σ reads

$$\mathcal{L}_N=\sum_{i=1}^N\bar\psi^i\left(\partial\!\!\!/+\lambda\sigma\right)\psi^i+\frac{1}{2}M\sigma^2. \tag{1}$$

This theory was constructed rigorously in [18], where the existence of an interacting UV fixed point was established. A detailed study can be found in [15]. There are two phases, and the chiral symmetry can be dynamically broken. The σ-field equation gives $\sigma=-\lambda\bar\psi\psi/M$, whence the name “composite boson” for σ.

The theory is well-defined also if we set $M=0$. The model

$$\mathcal{L}=\sum_{i=1}^N\bar\psi^i\left(\partial\!\!\!/+\lambda\sigma\right)\psi^i \tag{2}$$

is conformal both at the classical and quantum levels, as we now prove. We call it the σ_N conformal field theory. At the classical level no scale is present. The renormalized lagrangian has the form

$$\mathcal{L}=Z_\psi\bar\psi\partial\!\!\!/\psi+\lambda_{\rm B}Z_\sigma^{1/2}Z_\psi\bar\psi\sigma\psi.$$

The bare coupling can be written as $\lambda_{\rm B}=\lambda Z_\lambda\mu^{\varepsilon/2}$. The number of independent renormalization constants is equal to the number of independent fields and therefore we can interpret two Z's as the wave-function renormalization constants of ψ and σ, and set $Z_\lambda\equiv 1$. This ensures that $\beta_\lambda\equiv 0$ in $D=3$ and proves that the theory is conformal also at the quantum level. At the level of the trace anomaly, conformality (i.e. $\Theta\equiv 0$) follows from the fact that all local dimension-3 operators are proportional to the field equations.

The dynamical σ kinetic term is generated by the fermion bubble and gives the momentum space quantum kinetic action

$$\Gamma_{\rm kin}[\sigma]=\frac{1}{2}|\sigma(k)|^2\sqrt{k^2}, \tag{3}$$

having fixed the normalization with

$$\lambda^2 N = 8 + \mathcal{O}(1/N), \tag{4}$$

In x-space we have

$$\langle \sigma(x)\ \sigma(0) \rangle = \frac{1}{2\pi^2 |x|^2} \tag{5}$$

in the $N = \infty$ limit. At finite N a straightforward application of the Callan–Symanzik equations shows that the σ-two-point function has the form

$$\Gamma_{\sigma\sigma} = A(\lambda)\sqrt{k^2}\left(\frac{\mu^2}{k^2}\right)^{\gamma_\sigma(\lambda)}, \tag{6}$$

or, in x-space,

$$\langle \sigma(x)\ \sigma(0) \rangle = \frac{A'(\lambda)}{|x|^{2+2\gamma_\sigma(\lambda)}\mu^{2\gamma_\sigma(\lambda)}}. \tag{7}$$

The numerical coefficients $A(\lambda)$ and $A'(\lambda)$ do not have here a direct physical meaning, because they are scheme-dependent and can be changed by redefining μ.

Formulas (6) and (7) have the expected form for a conformal field theory. A non-vanishing anomalous dimension $\gamma_\sigma(\lambda)$ proves that the theory is interacting. We find [15, 19]

$$\gamma_\psi = \frac{2}{3N\pi^2}, \qquad \gamma_\sigma = -\frac{16}{3N\pi^2}. \tag{8}$$

Now, we want to define renormalization-group flows interpolating between the σ_{N+M} and the σ_N conformal field theories. Let us consider the lagrangian

$$\mathcal{L}_{NM} = \sum_{i=1}^{M} \bar{\chi}^i \left(\partial\!\!\!/ + g\sigma\right)\chi^i + \sum_{i=1}^{N} \bar{\psi}^i \left(\partial\!\!\!/ + \lambda\sigma\right)\psi^i. \tag{9}$$

Here we expand perturbatively in g, or actually $\bar{g} = g/\lambda$. For $\bar{g} = 0$ we have the σ_N model plus M free fermions. For $\bar{g} = 1$ we have the σ_{N+M} model. It is therefore natural to expect that the coupling $\bar{g}$ interpolates between the two fixed points. We can show that there is a non-trivial beta function by studying the first perturbative corrections. We combine the small-$\bar{g}$ expansion with the large-N expansion. We also assume that $\bar{g}^2 M/N \ll 1$. Since $\bar{g}$ varies from 0 to 1, this means that M is much smaller than N. The renormalized lagrangian reads

$$\mathcal{L}_R = \sum_{i=1}^{M} Z_\chi \bar{\chi}^i \left(\partial\!\!\!/ + gZ_{\bar{g}}Z_\sigma^{1/2}\sigma\right)\chi^i + \sum_{i=1}^{N} Z_\psi \bar{\psi}^i \left(\partial\!\!\!/ + \lambda Z_\sigma^{1/2}\sigma\right)\psi^i + \mathcal{L}_{\rm non\,loc}$$

and the evanescent, renormalization-group invariant, non-local kinetic term reads, in the general case:

$$\mathcal{L}_{\rm non\,loc} = \frac{1}{2}|\sigma(k)|^2\sqrt{k^2}\left[1 - \frac{\lambda_{\rm B}^2 N}{8}(k^2)^{-\varepsilon/2}\right]\exp\left(-2\int_{\ln\mu}^{\ln\sqrt{k^2}} \gamma_\sigma(\ln\mu')\,{\rm d}\ln\mu'\right).$$

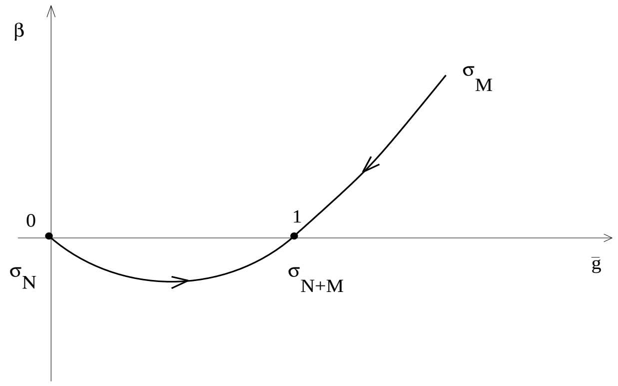

Figure 1: Beta function of dual RG flows.

From the results (8) we easily get, to the lowest order,

$$Z_{\psi}=1-\frac{4\mu^{-\varepsilon}}{3N\pi^{2}\varepsilon},\qquad Z_{\chi}=1-\frac{4\bar{g}^{2}\mu^{-\varepsilon}}{3N\pi^{2}\varepsilon},\qquad Z_{\sigma}=1+\frac{32\mu^{-\varepsilon}}{3N\pi^{2}\varepsilon},\qquad Z_{\bar{g}}=1+\frac{16\left(\bar{g}^{2}-1\right)\mu^{-\varepsilon}}{3N\pi^{2}\varepsilon}$$

We therefore obtain

$$\beta_{\bar{g}}=\frac{16}{3N\pi^{2}}\bar{g}\left(\bar{g}^{2}-1\right)+\mathcal{O}(\bar{g}/N^{2},\bar{g}^{5}/N) \tag{10}$$

and conclude that the σ_N model plus M decoupled fermions is the UV limit of the flow and the σ_{N+M} point is the IR limit. Remarkably, the first orders in $\bar{g}$ single out correctly both fixed points. This means that, presumably, every truncation of the perturbative expansion of $\beta_{\bar{g}}$ factorizes the expected $\bar{g}\left(\bar{g}^2-1\right)$. We show below that this is indeed the case. The theories with couplings $\bar{g}$ and $-\bar{g}$ are clearly equivalent.

The flows (9) satisfy a natural strong–weak coupling duality, associated with the replacement $\bar{g}\leftrightarrow 1/\bar{g}$, $N\leftrightarrow M$. The dual flow interpolates from the UV σ_M model with N free fermions to the IR σ^{N+M} model. Pairs of dual flows have the IR limits in common. Finally, the self-dual flow has $N=M$. We immediately realize that the σ_M model plus N free fermions is the fixed point at $\bar{g}=\infty$. It is natural to conjecture that the points $\bar{g}=0,1,\infty$ are all the fixed points of the exact beta function. The dual flows are plotted in Fig. 1.

The mentioned duality and fixed points are non-perturbative properties of the flows and are self-evident from the construction. We have already seen that, unexpectedly, the lowest order beta function (10), calculated for $\bar{g}\ll 1$, vanishes at $\bar{g}=1$. What is even more astonishing is that, with a little improvement, the beta function vanishes also at $\bar{g}=\infty$ and satisfies the mentioned duality exactly. To see this, let us relax the assumption $\bar{g}^2M/N\ll 1$, so that M and N can be of the same order. Diagram (a) is proportional to $N+M\bar{g}^2$. The above formulas can be corrected replacing N by $N+M\bar{g}^2$. In particular, the lowest-order beta function (10) becomes

$$\beta_{\bar{g}}=\frac{16}{3\pi^{2}}\frac{\bar{g}\left(\bar{g}^{2}-1\right)}{(N+M\bar{g}^{2})},$$

and does satisfy the $\bar{g} \leftrightarrow 1/\bar{g}$, $N \leftrightarrow M$ duality, because

$$\beta_{1/\bar{g}} = \frac{16}{3\pi^2} \frac{1/\bar{g}\,(1/\bar{g}^2 - 1)}{(M + N/\bar{g}^2)}.$$

The remarkable perturbative features that we have outlined are explained by an exact relation between the beta function and the anomalous dimension of σ, that we now derive. This formula is a sort of three-dimensional analogue of certain common formulas in four-dimensional supersymmetric theories, such as the NSVZ beta function [3], or the beta function of the superpotential coupling. We stress that in three dimensions we do not need supersymmetry for this.

We write the renormalized lagrangian in a manifestly dual form:

$$\begin{aligned} \mathcal{L}_R &= \sum_{i=1}^{M} V(g, M; \lambda, N; \varepsilon) \bar{\chi}^i \left(\partial\!\!\!/ + gU(g, M; \lambda, N; \varepsilon)\sigma \right) \chi^i \\ &\quad + \sum_{i=1}^{N} V(\lambda, N; g, M; \varepsilon) \bar{\psi}^i \left(\partial\!\!\!/ + \lambda U(\lambda, N; g, M; \varepsilon)\sigma \right) \psi^i. \end{aligned}$$

We have

$$\begin{aligned} Z_\chi &= V(g, M; \lambda, N; \varepsilon), \qquad Z_\psi = V(\lambda, N; g, M; \varepsilon), \\ Z_\sigma &= U^2(\lambda, N; g, M; \varepsilon), \qquad Z_g = \frac{U(g, M; \lambda, N; \varepsilon)}{U(\lambda, N; g, M; \varepsilon)}, \end{aligned}$$

and find

$$\beta_{\bar{g}} = \bar{g}\,(\gamma_\sigma - \tilde{\gamma}_\sigma) \equiv \bar{g}\left[\gamma_\sigma(\bar{g}, M; N) - \gamma_\sigma(1/\bar{g}, N; M)\right]. \tag{11}$$

Observe that $\gamma_\sigma(1, M; N) = \gamma_\sigma(M + N)$. We can immediately check the duality of the exact beta function:

$$\beta_{1/\bar{g}} = \frac{1}{\bar{g}}\left[\gamma_\sigma(1/\bar{g}, N; M) - \gamma_\sigma(\bar{g}, M; N)\right]. \tag{12}$$

The beta function vanishes at the fixed points $\bar{g} = 0, \infty$ and the solutions of

$$\gamma_\sigma(\bar{g}, M; N) = \gamma_\sigma(1/\bar{g}, N; M). \tag{13}$$

Using the fact that $\gamma_\sigma(1, M; N) = \gamma_\sigma(M + N)$ we know that $\bar{g} = 1$ is a solution. We expect that this is the unique solution of the condition (13).

The trace anomaly reads

$$\Theta = \beta_{\bar{g}} \sigma \sum_{i=1}^{M} \bar{\chi}^i \chi^i \equiv \beta_{\bar{g}} \mathcal{O}$$

and, correctly, does not vanish using the field equations.

More generally, we can consider the model

$$\mathcal{L} = \sum_{i=1}^{k} \sum_{j=1}^{N_i} V(\lambda_i, N_i; \lambda, N; \varepsilon) \bar{\psi}^j_{(i)} \left(\partial\!\!\!/ + \lambda_i U(\lambda_i, N_i; \lambda, N; \varepsilon)\sigma \right) \psi^j_{(i)},$$

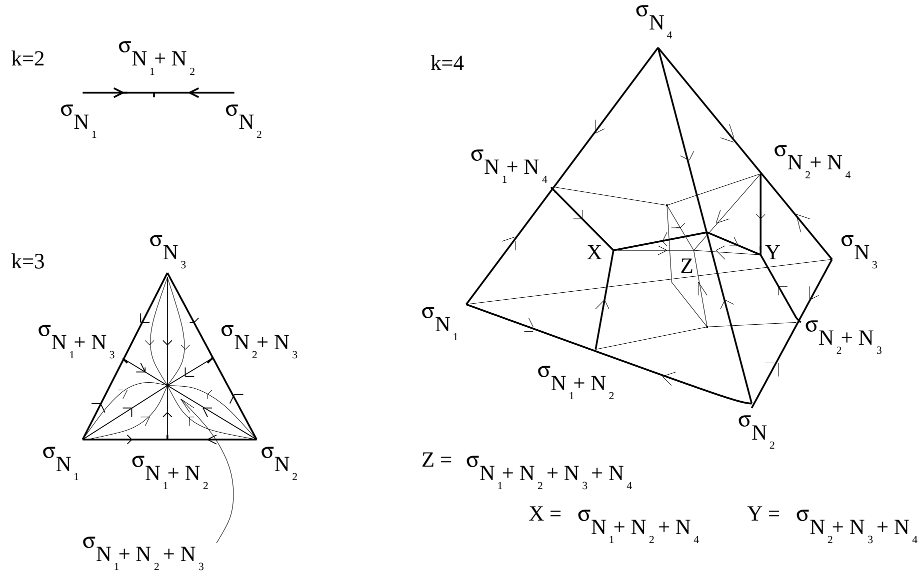

Figure 2: Polyhedrical RG patterns interpolating between pairs of σ_N models.

where the argument λ, N in $(\lambda_i, N_i; \lambda, N; \varepsilon)$ refers to the set of couples λ_j, N_j with $j \neq i$. Clearly, $V(\lambda_i, N_i; \lambda, N; \varepsilon)$ and $U(\lambda_i, N_i; \lambda, N; \varepsilon)$ are symmetric with respect to the exchanges $\lambda_j, N_j \leftrightarrow \lambda_l, N_l$ with $j, l \neq i$. Choosing λ_k to be of order unity and all the other λ's small, we have

$$\begin{aligned}
\beta_i &= \bar{\lambda}_i \left(\gamma_\sigma - \tilde{\gamma}_\sigma^{(i)}\right) \\
&\equiv \bar{\lambda}_i \left[\gamma_\sigma(\bar{\lambda}_1, N_1; \cdots \bar{\lambda}_i, N_i \cdots; \bar{\lambda}_{k-1}, N_{k-1}; N_k)\right. \\
&\quad \left. -\gamma_\sigma(\bar{\lambda}_1/\bar{\lambda}_i, N_1; \cdots 1/\bar{\lambda}_i, N_k \cdots; \bar{\lambda}_{k-1}/\bar{\lambda}_i, N_{k-1}; N_i)\right], \\
\gamma_\sigma &= \gamma_\sigma(\bar{\lambda}_1, N_1; \cdots; \bar{\lambda}_{k-1}, N_{k-1}; N_k) = \frac{\mathrm{d} \ln U(\lambda_k, N_k; \lambda, N; \varepsilon)}{\mathrm{d} \ln \mu},
\end{aligned}$$

where $\bar{\lambda}_i = \lambda_i/\lambda_k$ and $i = 1, \cdots, k-1$. The list of fixed points is obtained by assigning the values $0, 1, \infty$ to $\bar{\lambda}_1, \ldots, \bar{\lambda}_{k-1}$ in all possible ways, keeping in mind that when some $\bar{\lambda}$'s are infinite, it is immaterial whether the finite $\bar{\lambda}$'s are 0 or 1. In total, we have $2^k - 1$ fixed points, corresponding to the models $\sigma_{\sum_s N}$ for all possible subsets s of $(N_1, \ldots, N_k)$. The flows are naturally associated with the regular polyhedron having k faces in $k-1$ dimensions (triangle for $k = 3$, tetrahedron for $k = 4$, etc.) and the dualities are symmetries of this polyhedron. The RG patterns for $k = 2, 3, 4$ are illustrated in Fig. 2.

The trace anomaly reads

$$\Theta = \sigma \sum_{i=1}^{k-1} \beta_i \sum_{j=1}^{N_i} \bar{\psi}_{(i)}^j \psi_{(i)}^j.$$

Flows interpolating between the UV σ_{N+M} and IR σ_N fixed points can be obtained by giving mass to M fermions. For the general purposes mentioned in the introduction, these flows are less interesting than the pure RG flows, which preserve conformality at the classical level and run only due to the dynamical scale μ [5]. In some cases, nevertheless, such as the vector models constructed below, giving masses to the fermions seems the only simple way to interpolate between pairs of fixed points, because a non-renormalization theorem forbids the running of the gauge coupling constant.

Vector four-fermion models were also considered in [13]. Non-Abelian composite gauge bosons have a quantized and non-renormalized coupling constant. The Abelian coupling, instead, is non-renormalized, but can take arbitrary values. Conformal windows of the type constructed with the σ_N models cannot be constructed with vector models.

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