Compactified strings and Quantum Mechanics on the Jacobian torus

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Outline

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1. Introduction
2. Main correspondence
   - Free field theory on a Riemann surface
   - Quantum mechanics on the Jacobian torus
3. Dualities
   - T-duality vs high/low temperature duality
   - Target space vs worldsheet
   - Dimensionality vs topology
4. Conclusions

Main results

- Bosonic free field theory
- World-sheet: RS of genus g
- Target space: $\mathbb{S}^1$ ($\mathbb{T}^d$)
- Classical action of an instantonic solution

- QM of a free particle
- Space: $2g$ dimensional torus
- Finite temperature
- Eigenvalue of the Laplacian

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$$S_{mn} = \lambda_{mn}$$

$$\downarrow$$

$$Z_{cl}(\beta) = Z_{stat}(\beta)$$

Motivations

Immediate consequences

- T-duality and High/Low Temperature duality
- Worldsheet/Target Space duality [Giveon, Porrati, Rabinovici '94]

Related topics

- Dimensionality vs topology and negative curvature target space [Silverstein '05]
- Spectral problem on Riemann surfaces
- KMS states and non-commutative geometry [Connes, Marcolli, Ramachandran '05]

Outline

Definitions

Notation

- Riemann surface Σ of genus g
- Symplectic basis of 1-cycles $\{\alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g\}$ with

$$\#(\alpha_i,\alpha_j)=0=\#(\beta_i,\beta_j)\;,\qquad \#(\alpha_i,\beta_j)=\delta_{ij}$$

- Normalized basis of holomorphic 1-differentials

$$\int_{\alpha_i}\omega_j=\delta_{ij}\;,\qquad \Omega_{ij}=\int_{\beta_i}\omega_j$$

Definitions

Free field X on Σ with

- Target space $\mathbb{S}^1 = \mathbb{R}/2\pi R\mathbb{Z}$
- Action $S[X] = \frac{1}{4\pi R^2} \int_\Sigma \partial X \bar{\partial} X$
- Partition function $Z(\beta) = \int_{(\Sigma, \mathbb{S}^1)} dX\, e^{-\beta S[X]}$
- Bosonic strings $\beta = R^2/\alpha'$

Classical solutions

Splitting

$$X = X^{cl} + X^{q}$$

Equation of motion

$$\partial\bar{\partial}X^{cl} = 0$$

Multivaluedness

$$X^{cl}_{mn}(z + p\cdot\alpha + q\cdot\beta) = X^{cl}_{mn}(z) + 2\pi R(m\cdot q - n\cdot p)$$

$$\downarrow$$

Solution

$$X^{cl}_{mn}(z) = \frac{\pi R}{i}\Big[(m + n\bar{\Omega})(\mathrm{Im}\,\Omega)^{-1}\int^{z}\omega - c.c.\Big]\,, \qquad m,n\in\mathbb{Z}^{g}$$

Partition function

$$S[X] = S[X^{cl}] + S[X_q]$$

$$\downarrow$$

$$Z(\beta) = Z_{cl}(\beta) Z_q(\beta)$$

$$S_{mn} \equiv S[X^{cl}_{mn}] = \pi(m + n\bar{\Omega})(\mathrm{Im}\,\Omega)^{-1}(m + \Omega n)$$

$$Z_{cl}(\beta) = \sum_{m,n \in \mathbb{Z}^g} e^{-\beta S_{mn}}$$

The complex torus J_Ω

$$J_\Omega = \mathbb{C}^g / (\mathbb{Z}^g + \Omega \mathbb{Z}^g)$$

- Hodge metric on J_Ω $\quad ds^2 = \sum (2\,\mathrm{Im}\,\Omega)^{-1}_{ij} dz_i d\bar{z}_j$
- Laplacian on J_Ω $\quad \Delta = \sum (2\,\mathrm{Im}\,\Omega)_{ij} \frac{\partial}{\partial z_i} \frac{\partial}{\partial \bar{z}_j}$
- Hamiltonian $\quad H = \frac{\Delta}{2\pi}$

Partition function

$$Z_{stat}(\beta) = \mathrm{Tr}\, e^{-\beta H}, \qquad \beta = \frac{1}{kT}$$

Statistical partition function

- Eigenfunctions $\psi_{m,n} = \exp \pi[(m + n\bar{\Omega})(\operatorname{Im}\Omega^{-1})z - c.c.]$
- Eigenvalues $\lambda_{m,n} = \pi(m + n\bar{\Omega})(\operatorname{Im}\Omega^{-1})(m + \Omega n)$

$$\downarrow$$

$$\lambda_{mn} = S_{mn}$$

$$\downarrow$$

$$Z_{stat}(\beta) = \sum_{m,n\in\mathbb{Z}^g} e^{-\beta\lambda_{mn}} = \sum_{m,n\in\mathbb{Z}^g} e^{-\beta S_{mn}} = Z_{cl}(\beta)$$

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T-duality vs high/low temperature duality

Duality relation

$$Z(\beta) = \beta^{-g} Z(1/\beta)$$

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Free field theory on Σ

QM on Jacobian torus

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$$R \to R_{sd}/R$$

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T-duality	$\longleftrightarrow$	High-Low T duality

Toroidal compactification

- Free FT on Σ_Ω $\quad\longrightarrow\quad$ $\mathbb{T} \equiv \mathbb{R}^d/\Lambda$, with metric G_{ab}

$$S_{mn} = \pi G_{ab}(m^a + n^a\bar{\Omega})(\operatorname{Im}\Omega)^{-1}(m^b + \Omega n^b)$$

where $m, n \in \Lambda \otimes \mathbb{Z}^g$

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- QM on $J_\Omega\otimes\mathbb{T}^*=\mathbb{C}^{gd}/\Lambda_\Omega\otimes\Lambda^*$

Laplacian $\quad \Delta=G^{-1}_{ab}(2\operatorname{Im}\Omega)^{-1}_{ij}\frac{\partial}{\partial z^{ai}}\frac{\partial}{\partial\bar{z}^{bj}}$

Eigenfunctions $\quad \psi_{mn}=\exp\pi[G_{ab}(m^a+n^a\bar{\Omega})(\operatorname{Im}\Omega)^{-1}z^b-c.c.]$

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$$S_{mn} = \pi G_{ab}(m^a + n^a\bar{\Omega})(\operatorname{Im}\Omega)^{-1}(m^b + \Omega n^b) = \lambda_{mn}$$

where $m, n \in \Lambda \otimes \mathbb{Z}^g$

- QM on $J_\Omega \otimes \mathbb{T}^* = \mathbb{C}^{gd}/\Lambda_\Omega \otimes \Lambda^*$

Laplacian $\quad \Delta = G_{ab}^{-1}(2\operatorname{Im}\Omega)_{ij}^{-1}\frac{\partial}{\partial z^{ai}}\frac{\partial}{\partial \bar{z}^{bj}}$

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Target space vs worldsheet

FT on Σ_Ω (genus g) $\quad\longrightarrow\quad$ J_τ ($2h$-dim) with metric $(\mathrm{Im}\,\tau)^{-1}$

$$\updownarrow$$

QM on $\mathbb{R}^{4gh}/\Lambda_\Omega \otimes \Lambda_\tau$ with metric $\frac{1}{2}(\mathrm{Im}\,\Omega)^{-1} \otimes (\mathrm{Im}\,\tau)^{-1} \otimes \mathbf{1}_4$

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Target-space $\leftrightarrow$ Worldsheet [Giveon, Porrati, Rabinovici '94]

Compactification on Riemann surfaces

- Target space $\mathbb{R}^D$
- Density of states

$$\rho(m) \sim e^{A\sqrt{\alpha'}m\sqrt{D-2}}$$

- Target space Σ
- Density of (winding) states

$$\rho(m) \sim e^{\sqrt{\alpha'}m\sqrt{\frac{(2g-2)l_s^2}{V_\Sigma}}}$$

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In the limit $V_\Sigma \sim l_s^2$

$$D = 2g$$

$$\downarrow$$

Dimensionality $\leftrightarrow$ Topology

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Conjecture

Target space is the Jacobian torus [McGreevy, Silverstein, Starr '06]

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Conclusions

- 2D bosonic free FT $\longleftrightarrow$ QM on higher dim. space
- Instantonic action $\longleftrightarrow$ Energy eigenvalue
- Compactification radius $\longleftrightarrow$ Temperature
- T-duality $\longleftrightarrow$ High/Low Temperature duality
- Target Space vs World sheet duality
- Topology $\longleftrightarrow$ Dimensionality

References

- M. Matone, P. Pasti, S. Shadchin and R. Volpato, Phys. Rev. Lett. **97** (2006) 261601 [arXiv:hep-th/0607133]
- M. Matone, P. Pasti and R. Volpato, work in progress.