Self-organized criticality in nature

Giuseppe Clemente
giuseppe.clemente@pi.infn.it

Istituto Nazionale di Fisica Nucleare

Università di Pisa

Pisa, 28 June 2018

Overview

A world of power-laws $f(x) = C\, x^{-\alpha}$

Power-laws occur in many contexts:

- wealth among individuals (Pareto's law or 80/20 rule)
- city sizes, family names, words frequency (Zipf's law)
- physical stimulus vs perceived intensity (Steven's law)
- frequency of publications per author (Lotka's law)
- earthquake magnitude vs frequency (Gutenberg-Richter's law)
- etc ...

Power-laws $\implies$ scale-invariance: $f(\lambda x) \propto f(x)$

Purpose of self-organized critical (SOC) models

Provide <u>robust</u> mechanisms giving rise to <u>scale invariant</u> behaviors

Requirements for self-organized criticality

Assumptions

- Observed data produced by an underlying dynamical process
- Complex dynamics, involving interactions between subsystems

Required feature: "Adaptation to the edge of chaos"

- robustness: existence of an attractor for the dynamics.
- scale-invariance: as self-similarity in chaotic systems

SOC systems are dynamically self-driven toward their critical point
$\Longrightarrow$ the same critical behavior is reached without fine-tunings.

Why criticality is useful

- The typical exponential behavior of observables turns into power-laws (diverging correlation length ξ).
- **Universality** $\implies$ simple models can be effective in describing different phenomena regardless of details of the system.
- The exponents of power-laws are universal, depending only on very general features of the system (i.e. dimensionality).

A good model for a critical phenomenon should be simple enough to be amenable, and at the same time accurate enough to reproduce correctly all critical exponents.

General features of SOC models

Features common to all SOC models are:

- Presence of a critical threshold which induces nonlinear energy dissipation events (avalanches), driven by positive feedback
- Continuous input of energy in the system, usually at slow rates with respect to avalanche durations
- The energy input is delivered randomly in space and/or time (i.e. as a Poisson process)

Lack of guarantees

The features above may not always lead to SOC behaviour
(an example discussed later)

Historic example: (abelian) sandpile BTW model

PHYSICAL REVIEW LETTERS

VOLUME 59 | 27 JULY 1987 | NUMBER 4

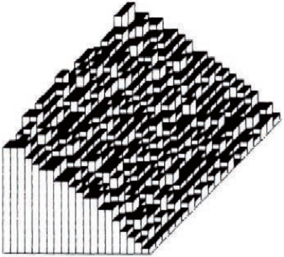
Self-Organized Criticality: An Explanation of 1/*f* Noise

Per Bak, Chao Tang, and Kurt Wiesenfeld

Physics Department, Brookhaven National Laboratory, Upton, New York 11973

(Received 13 March 1987)

We show that dynamical systems with spatial degrees of freedom naturally evolve into a self-organized critical point. Flicker noise, or 1/*f* noise, can be identified with the dynamics of the critical state. This picture also yields insight into the origin of fractal objects.



Cellular automaton for 2D abelian sandpile (Bak et al. [BTW87]):
local height: $h_{x,y}$, concavity $z_{x,y}$

$$z_{x,y} = \Delta h_{x,y} = 4h_{x,y} - h_{x+1,y} - h_{x,y+1} - h_{x-1,y} - h_{x,y-1}.$$

toppling if $z_{x,y} \geq z_c$ (usually $z_c = 4$):

- $z_{x,y} \to z_{x,y} - 4$,
- $z_{x\pm1,y} \to z_{x\pm1,y} + 1$,
- $z_{x,y\pm1} \to z_{x,y\pm1} + 1$

Other simple example: forest-fire/epidemic model

First formulation of the Forest Fire Model (FFM) given by Bak et al. [BCT90].

Three-state cellular automaton on an hypercubic lattice with L^d sites with the following update rules:

(i) A burning tree becomes an empty site.

(ii) A green tree becomes a burning tree if at least one of its nearest neighbors is burning.

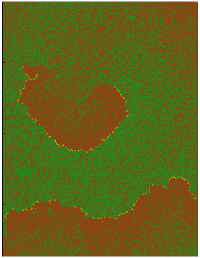
(iii) At an empty site a tree grows with probability p.

Bak's FFM: critiques about self-organized behaviour

Numerical simulations by Grassberger have shown that:

Bak's FFM is not critical but deterministic

- oscillatory behaviour
- spiral shaped firefronts
- characteristic scale: $1/p$



Deterministic behavior is common to many cellular automata, i.e. the famous Conway's game of life.

The Drossel-Schwabl FFM I

Drossel and Schwabl proposed a modification of the Bak's model introducing a fourth updating rule [CDS96]:

(iv) A tree without a burning nearest neighbor becomes a burning tree during one time step with probability f.

For sufficiently small birth-rate p, this model is shown to be critical with a relevant parameter (f/p) and critical point at $(f/p) \to 0$.

p/f is a measure for the number of trees growing between two lightning strokes, $T(s_{max})$ is the max time interval for burning a cluster. Conditions for SOC behaviour:

$$T(s_{max}) \ll p^{-1} \ll f^{-1},$$

double separation of time scales.

The Drossel-Schwabl FFM II

Lets define some observables with scaling laws (in a finite system):

- $\bar{s} \propto (f/p)^{-\lambda}$, mean cluster size,
- $T(\bar{s}) \propto \xi^{z}$, mean time interval for burning a cluster,
- $N(s) \propto s^{-\tau}\,\Phi(s/s_{\xi})$, mean number of s-clusters ($\tau > 2$),
- $R(s) \propto s^{1/\mu}\,\widetilde{\Phi}(s/s_{\xi})$, mean gyration radius for an s-cluster,
- $\xi \equiv R(\bar{s}) \propto (f/p)^{-\nu}$, correlation length.

Where $s_{\xi} \propto \bar{s} \propto (f/p)^{-\lambda}$ is a cutoff cluster size, and $\Phi, \widetilde{\Phi}$ are universal functions ($\Phi(x) \to 1$ for $x \ll 1$, $\Phi(x) \to 0$ for $x \gg 1$).

The Drossel-Schwabl FFM III

Assuming $p \gg T(s_{max})^{-1}$ we can give an estimate of $\overline{s}$:

$$\overline{s} = \frac{1-\rho_t}{\rho_t}\,(f/p)^{-1},$$

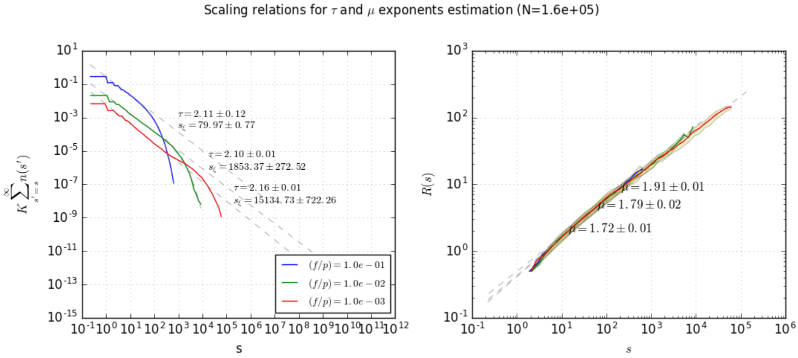
where ρ_t is the mean tree-density; this implies $\lambda = 1$.
Finally, using the scaling relations:

$$\lambda = \nu\mu, d = \mu(\tau - 1)$$

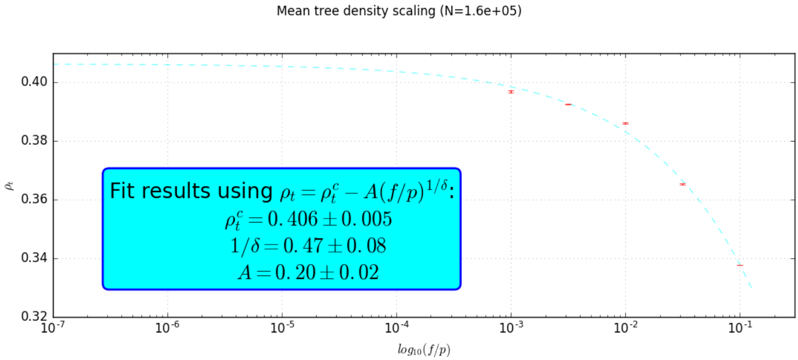
We obtain the following estimate of the critical exponents:

$$\lambda = 1, \tau = 2, \mu = d, \nu = 1/d$$

Simulation of the Drossel-Schwabl FFM I



Simulation of the Drossel-Schwabl FFM II



Simulation of the Drossel-Schwabl FFM III

Results of numerical simulations:

Drossel-Schwabl [CDS96]:	**Ours**:
$\tau \approx 2.15$	$\tau = 2.16 \pm 0.01$
$\mu \approx 1.96$	$\mu = 1.91 \pm 0.01$
$\rho_t^c \approx 0.41$	$\rho_t^c = 0.406 \pm 0.005$
$1/\delta \approx 0.5$	$1/\delta = 0.47 \pm 0.08$

Fractal-diffusive SOC models

It is useful to extract expectations for a general class of SOC models in order to compare with experimental data.

Observing the time evolution of avalanche volumes it is possible to measure the fractal dimension D_S, defined by $V(t) \propto t^{D_S/2}$.
A coarsest-order analytical estimate of D_S is the average between the minimum and the maximum dimension, i.e. $D_S \simeq \frac{(1+d)}{2}$.

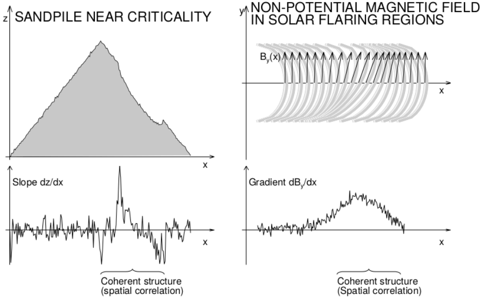
Since the probability for an avalanche with size V is $N(V) \propto \frac{V_{tot}}{V} \propto V^{-1}$ and the flux $\frac{dE(t)}{dt} \propto \langle E \rangle V_S(t) \propto t^{D_S/2}$, an estimate of the total dissipated energy goes like:

$$E(t) = \int_0^t d\tau \frac{dE(\tau)}{d\tau} \propto t^{1+D_S/2}$$

$$\implies N(E)dE = N(T(E))|\frac{dT}{dE}|dE \propto E^{-[1+(d-1)/(D_S+2)]} \equiv E^{-\alpha_E}.$$

A naive expectation for $d = 3$ gives the critical exponent $\alpha_E \simeq 1.5$.

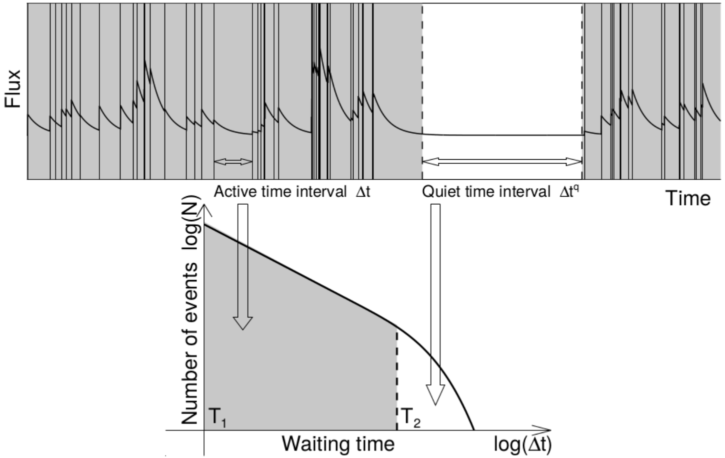
Solar flares



SOC features in solar flares

- threshold: critical stressing angle between a potential and non-potential magnetic field line in an active region
- avalanche: magnetic flux reconnection triggering the flare
- dissipated energy: measured by the variation of magnetic energy before and after the flare, or by the plasma thermal energy, or by the kinetic energy of accelerated particles.

Solar flares



Most of the data collected in form of the energy flux observations.

Solar flares: hard X-rays

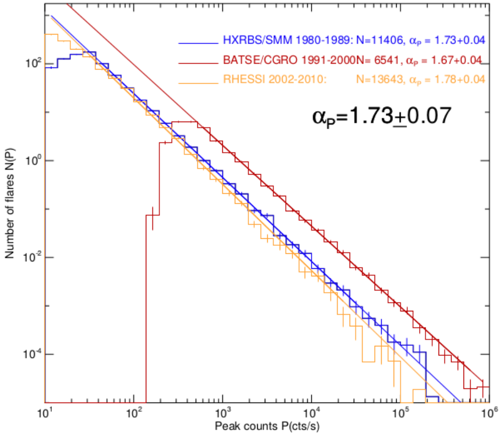


Fig. 9.— Occurrence frequency distributions of hard X-ray peak count rates P [cts s^{-1}] observed with HXRBS/SMM (1980 – 1989), BATSE (1991 – 2000), and RHESSI (2002 – 2010), with powerlaw fits. An average pre-flare background of 40 [cts s^{-1}] was subtracted from the HXRBS count rates. Note that BATSE/CGRO has larger detector areas, and thus records higher count rates (Aschwanden 2011b).

Seemingly good power-law behaviour!

Comparison between FD-SOC predictionand experiments

Table 2: Frequency distributions measured from solar flares in hard X-rays and γ-rays. The prediction is based on the FD-SOC model (Aschwanden 2012a).

Powerlaw slope of peak flux α_P	Powerlaw slope of fluence α_E	Powerlaw slope of durations α_T	Number of events n	Instrument and threshold energy	References
1.8			123	OSO–7(>20 keV)	Datlowe et al. (1974)
2.0			25	UCB(>20 keV)	Lin et al. (1984)
1.8			6775	HXRBS(>20 keV)	Dennis (1985)
1.73±0.01			12,500	HXRBS(>25 keV)	Schwartz et al. (1992)
1.73±0.01	1.53±0.02	2.17±0.05	7045	HXRBS(>25 keV)	Crosby et al. (1993)
1.71±0.04	1.51±0.04	1.95±0.09	1008	HXRBS(>25 keV)	Crosby et al. (1993)
1.68±0.07	1.48±0.02	2.22±0.13	545	HXRBS(>25 keV)	Crosby et al. (1993)
1.67±0.03	1.53±0.02	1.99±0.06	3874	HXRBS(>25 keV)	Crosby et al. (1993)
1.61±0.03			1263	BATSE(>25 keV)	Schwartz et al. (1992)
1.75±0.02			2156	BATSE(>25 keV)	Biesecker et al. (1993)
1.79±0.04			1358	BATSE(>25 keV)	Biesecker et al. (1994)
1.59±0.02		2.28±0.08	1546	WATCH(>10 keV)	Crosby (1996)
1.86	1.51	1.88	4356	ISEE–3(>25 keV)	Lu et al. (1993)
1.75	1.62	2.73	4356	ISEE–3(>25 keV)	Lee et al. (1993)
1.86±0.01	1.74±0.04	2.40±0.04	3468	ISEE–3(>25 keV)	Bromund et al. (1995)
1.80±0.01	1.39±0.01		110	PHEBUS(>100 keV)	Perez-Enriquez & Miroshnichenko (1999)
1.80±0.02		2.2±1.4	2759	RHESSI(>12 keV)	Su et al. (2006)
1.58±0.02	1.7±0.1	2.2±0.2	4241	RHESSI(>12 keV)	Christe et al. (2008)
1.6			243	BATSE(>8 keV)	Lin et al. (2001)
1.61±0.04			59	ULYSSES(>25 keV)	Tranquille et al. (2009)
1.73±0.07	1.62±0.12	1.99±0.35		Average	All HXR observations
1.67	**1.50**	**2.00**		FD-SOC prediction	Aschwanden (2012a)

FD-SOC prediction somewhat good

Comments and conclusions

Some critique is in order:

- It is still debatable if and when power-laws observed for various complex dynamical systems come by SOC mechanisms.
- Furthermore, power-laws can also be mimicked by log-normal distributions with large variance (multiplicative processes).
- Experimental data need large statistics to assess simple models (since computations of critical exponents are expensive).

Nevertheless, SOC models are still actively investigated in many fields, as earthquakes, neural networks, solar flares, giving often qualitatively reasonable results from elementary models.

Thank you for the attention!

References

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